SPECIAL PROJECT PROGRESS REPORT

All the following mandatory information needs to be provided. The length should *reflect the complexity and duration* of the project.

Reporting year	**2022**………………………………………………………
Project Title:	**Role of finite non-gaussianity in the evolution of wind wave fields, with applications to freak wave prediction**
Computer Project Account:	**SPGBSHRI** ………………………………………………
Principal Investigator(s):	**Prof V.I. Shrira** ………………………………………… ………………………………………………………………
Affiliation:	**School of Computing and Mathematics, Keele University, Keele ST5 5BG UK** …………………………
Name of ECMWF scientist(s) collaborating to the project (if applicable)	……………………………………………………………… ………………………………………………………………
Start date of the project:	**2022**………………………………………………………
Expected end date:	**2024**………………………………………………………

Computer resources allocated/used for the current year and the previous one

(if applicable)

Please answer for all project resources

		Previous year		Current year	
		Allocated	Used	Allocated	Used
High Performance Computing Facility	(units)	**N/A**	**N/A**	**1000000**	**20000**
Data storage capacity	(Gbytes)	**N/A**	**N/A**	**100**	**100**

Summary of project objectives (10 lines max)

1. To create, for the first time, the numerical model of wind waves within the kinetic theory with the account for weak nonlinearity and weak non-gaussianity.

2. To perform direct comparisons with the DNS

3. To examine implications for the probability of freak waves

4. To get new insights into the input and dissipation functions

5. To formulate recommendations for wind wave modelling

Summary of problems encountered (10 lines max)

No particular problems encountered …………………………………………………………………....

Summary of plans for the continuation of the project (10 lines max)

1. To finish the study of the discrete case, with the full solution of the system of equations for evolution of correlators and wave amplitudes.
2. To create the numerical model for continuous wave fields, extending the existing code for the generalised kinetic equation
3. To perform detailed comparison with the DNS results, both with and without wind forcing
4. To perform a direct comparison of the evolution of higher statistical moments

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List of publications/reports from the project with complete references

1. Annenkov S.Y., Shrira V.I. Effects of finite non-gaussianity on evolution of a random wind wave field. Phys Rev E. 2022 (Submitted)
2. Annenkov S.Y., Shrira V.I. New wave kinetic equation with the account for finite non-gaussianity. The 28th WISE meeting, Brest, France, 29 May – 2 June 2022

Summary of results

If submitted **during the first project year**, please summarise the results achieved during the period from the project start to June of the current year. A few paragraphs might be sufficient. If submitted **during the second project year**, this summary should be more detailed and cover the period from the project start. The length, at most 8 pages, should reflect the complexity of the project. Alternatively, it could be replaced by a short summary plus an existing scientific report on the project attached to this document. If submitted **during the third project year**, please summarise the results achieved during the period from July of the previous year to June of the current year. A few paragraphs might be sufficient.

The project was started in 2022. The central idea behind the project is the hypothesis that the neglect of finite non-gaussianity effects in the derivation of the Hasselmann kinetic equation leads to the discrepancy in spectral shapes, demonstrated by the comparison of simulations based on the Hasselmann kinetic equation (and also generalised kinetic equation) with direct numerical simulations, performed during the previous Special Project (2019-2021). Specifically, the "full" kinetic theory, without any additional approximations, leads to a pair of equations (Zakharov 1999)

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123} J_{0123} \delta_{0+1-2-3}\, d\mathbf{k}_{123}, \qquad (1)$$

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right) J_{0123} = 2\mathrm{i} T_{0123} f_{0123} + 2i\left(\hat{L}J\right)_{0123}, \qquad (2)$$

where $\mathbf{k}$ is a wavenumber and $\omega = \omega(\mathbf{k})$ is wave frequency, $n_j = n(\mathbf{k}_j)$ is the spectral amplitude, $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, T_{0123} is the interaction coefficient, $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, J_{0123} is the 4$^{\text{th}}$ order cumulant, and $\hat{L}J$ is a linear operator

$$\left(\hat{L}J\right)_{0123} = M_{0123} + M_{1023} - M_{2301} - M_{3201},$$

$$M_{0123} = n_1 \int T_{0145} J_{4523} \delta_{0145} \mathrm{d}\boldsymbol{k}_{45} - n_2 \int T_{0425} J_{1534} \delta_{0425} \mathrm{d}\boldsymbol{k}_{45} - n_3 \int T_{0435} J_{1524} \delta_{0435} \mathrm{d}\boldsymbol{k}_{45}.$$

The standard kinetic theory is obtained by neglecting $\hat{L}J$. Then the equation for J_{0123} can be solved as

$$J_{0123}(t) = 2\mathrm{i} T_{0123} \int_0^t e^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123} \mathrm{d}\tau + J_{0123}(0) e^{\mathrm{i}\Delta\omega t} \qquad (3)$$

(Annenkov & Shrira 2006), and the equation for n_0 becomes

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int \left\{ T_{0123}^2 \left[\int_0^t e^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123} \mathrm{d}\tau\right] - \frac{\mathrm{i}}{2} T_{0123} J_{0123}(0) e^{\Delta\omega t} \right\} \delta_{0+1-2-3} \mathrm{d}\boldsymbol{k}_{123},$$

which is the generalised kinetic equation. Taking the large-time limit, we recover the Hasselmann equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2\, f_{0123} \delta_{0+1-2-3} \delta(\Delta\omega) \mathrm{d}\mathbf{k}_{0123}.$$

But can the operator $\hat{L}J$ be safely dropped? The answer is not clear *a priori*. The operator represents the next-order correction, but as it stands on the right-hand side of the evolution equation for correlators, in the long term it may well play a role. Moreover, in that equation, we have spectral amplitudes n_j as coefficients. They become much larger around the spectral peak. Strictly speaking, this term is dropped just because the problem with it is far too complicated, and a kinetic equation in a closed form, describing evolution of spectrum in terms of spectrum, cannot be obtained. Meanwhile, neglecting $\hat{L}J$ means omitting the effects of finite non-gaussianity (incidentally, this is seen from the fact that the Hasselmann equation can be derived assuming random phases (Onorato & Dematteis 2020). The problem, however, is that if finite non-gaussianity is neglected, we cannot hope to to take finite nonlinearity effects into account either.

To get some idea about the significance of finite non-gaussianity term, it makes sense to consider a simple test problem. We build a *discrete* wave system which mimics the growth of an initially small harmonic on the spectral front. The target is to build a discrete system with a relatively small number of harmonics (dozens), well linked by nonlinear interactions and showing slow evolution of averaged amplitudes, so that the non-gaussianity remains low. An example of such a system is shown in figure 1. Evolution (simulated with the Zakharov equation, with averaging over 10000 realisations) is characterised by slow (over thousands of wave periods) growth of the lowest-frequency harmonic.

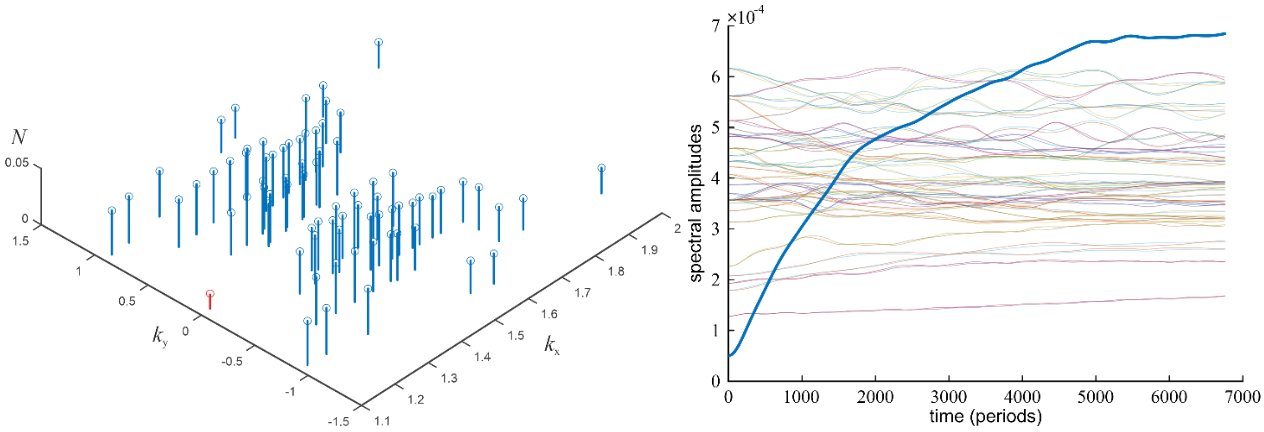

Figure 1. Left panel: Initial condition (spectral amplitudes vs wavevector components) for the discrete system. The lowest-frequency harmonic is shown in red. Right panel: Evolution of amplitudes vs time, with averaging over 10000 realisations. The amplitude of the lowest-frequency harmonic is shown by thick curve

The target is to try and recreate the growth rate using the equation for correlators (2), with and without the non-gaussianity terms. First, we solve the differential equation for correlators (2), and then obtain the growth rates using (1). Results are shown in figure 2.

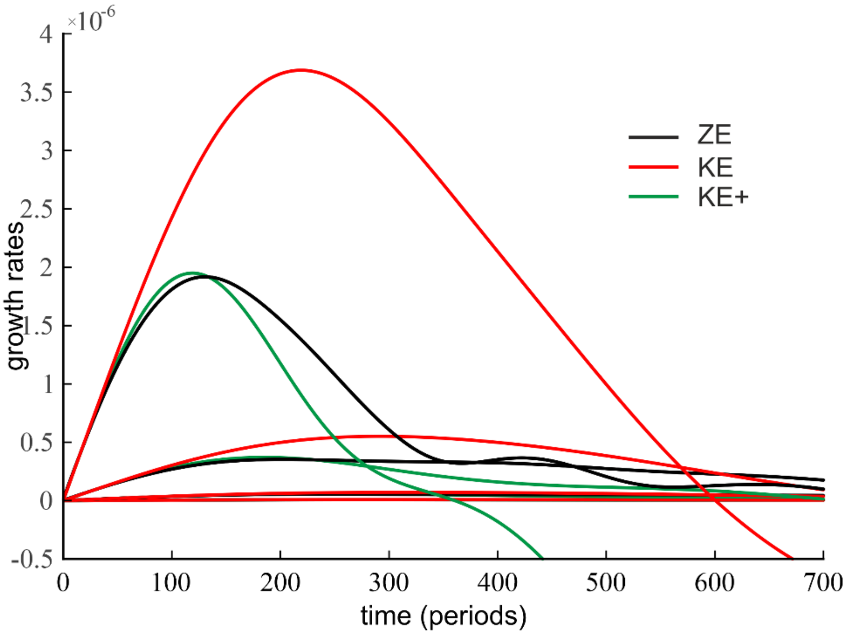

Figure 2. Growth rates for the lowest-frequency harmonic, obtained by numerical solution of Eq. (2) for correlators, using wave amplitudes obtained by numerical solution of the Zakharov equation. Four cases are shown together, for initial wave steepness of the total system equal to 0.035, 0.05, 0.071 and 0.1. Black curve: "true" growth rates from the Zakharov equation. Green curve: solution for the growth rate via correlators, with the account for non-gaussianity effects. Red curve: solution with non-gaussianity terms dropped

We plot growth rates for four cases, different only by a multiplier of all initial amplitudes, thus corresponding to different levels of nonlinearity between $\varepsilon = 0.035$ and $\varepsilon = 0.1$, where ε is the total initial wave steepness. Larger initial steepness corresponds to larger absolute growth rates.

Thus, we have found that the always discarded $\hat{L}J$ term is indeed not important for very low nonlinearity only. For larger but still small nonlinearity, the term becomes significant, and for $O(0.1)$ level of nonlinearity, discarding it leads to order 1 error in growth rate. In fact, it turns out that getting a closed kinetic equation for spectra comes at a price. The price is the loss of finite non-gaussianity effects, even though the Hasselmann equation corresponds to the second-order approximation in nonlinearity. Although non-gaussianity is weak, neglecting it violates the cornerstone principle of wave turbulence: equal play of weak nonlinearity and weak non-gaussianity. Ignoring finite non-gaussianity, we lose finite nonlinearity either. This confirms our initial hypothesis about the importance of finite non-gaussianity effects for wave kinetics.